ABSTRACT

SCATTERING AND TRANSMISSION MATRICES FOR THE FABRY-PEROT INTERFEROMETER

by

Curtis L. Harrington

Fabry-Perot interferometry concerns light passage through two parallel semi-mirrored surfaces, and the attenuation and transmission of the light based upon its wavelength in relationship to the separation of the semi-mirrored surfaces. Environmental effects which change the separation between the two parallel semi-mirrored surfaces can be measured indirectly by observing the characteristic frequencies of light passing through the interferometer.

For a constant frequency of light energy introduced into the Fabry-Perot interferometer, the mirror separation is measured by detecting the amount of light energy transmitted through or returned back in the direction of the source. Energy differences of a small magnitude require more sophisticated detectors to measure accurately, therefore a sensing interferometer should be able to exhibit significant energy changes as the quantities it is to measure change. For ease of routing an interferometer system, the case where a single optic fiber is used to transmit light into, and receive light from the Fabry-Perot interferometer along the same path, is of interest.

The transmission matrix, derivable from the scattering matrix, is most useful in the characterization of the interferometric sensor, and has physical meaning with respect to the processes occurring in the sensor. In addition, the transmission and scattering matrix characterizations illustrate the energy and phase relationships of the interferometer without regard to the types of materials employed in its construction.

WE, THE UNDERSIGNED MEMBERS OF THE COMMITTEE,

HAVE APPROVED THIS THESIS

SCATTERING AND TRANSMISSION MATRICES FOR THE

FABRY-PEROT INTERFEROMETER

by

Curtis L. Harrington

COMMITTEE MEMBERS

Kenneth James

Kenneth James, Ph.D. (chair) — Electrical Engineering

Michael Hassul

Michael Hassul, Ph.D. — Electrical Engineering

Fred Hadaegh

Fred Hadaegh, Ph.D. — Electrical Engineering

ACCEPTED AND APPROVED ON BEHALF OF THE UNIVERSITY

Mihir Kumar Das.

Mihir Das, Ph.D.
Associate Dean for Instruction
School of Engineering

California State University, Long Beach

December 1990

SCATTERING AND TRANSMISSION MATRICES FOR THE FABRY-PEROT INTERFEROMETER

A THESIS

Presented to the Department of Electrical Engineering

California State University, Long Beach

In Partial Fulfillment

of the Requirements for the Degree of

Master of Science

by

Curtis L. Harrington

December 1990

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION TO THE FABRY-PEROT

A plane mirror is defined as a material having an extremely smooth surface which can both transmit and reflect optical energy. In its simplest form, it can be considered much like a glass window which reflects a portion of the light incident upon it, and transmits the remainder. Typically the mirror used will have a coating to increase its reflectivity.

Of particular interest is a system whereby two partially transmitting mirrors are placed a finite distance apart. Light introduced through these mirrors is attenuated based upon the frequency components within the light beam. The operation of the Fabry-Perot interferometer is the most exemplary example of such a system.

The Fabry-Perot interferometer generally allows light waves having a wavelength characteristic which is some multiple of the distance between the plane mirrors to propagate completely through the two-mirror system. Light waves having a wavelength characteristic not a multiple of the mirror spacing are significantly attenuated. The space of separation between the two plane mirrors, often referred to as a resonant cavity, acts as a filter.

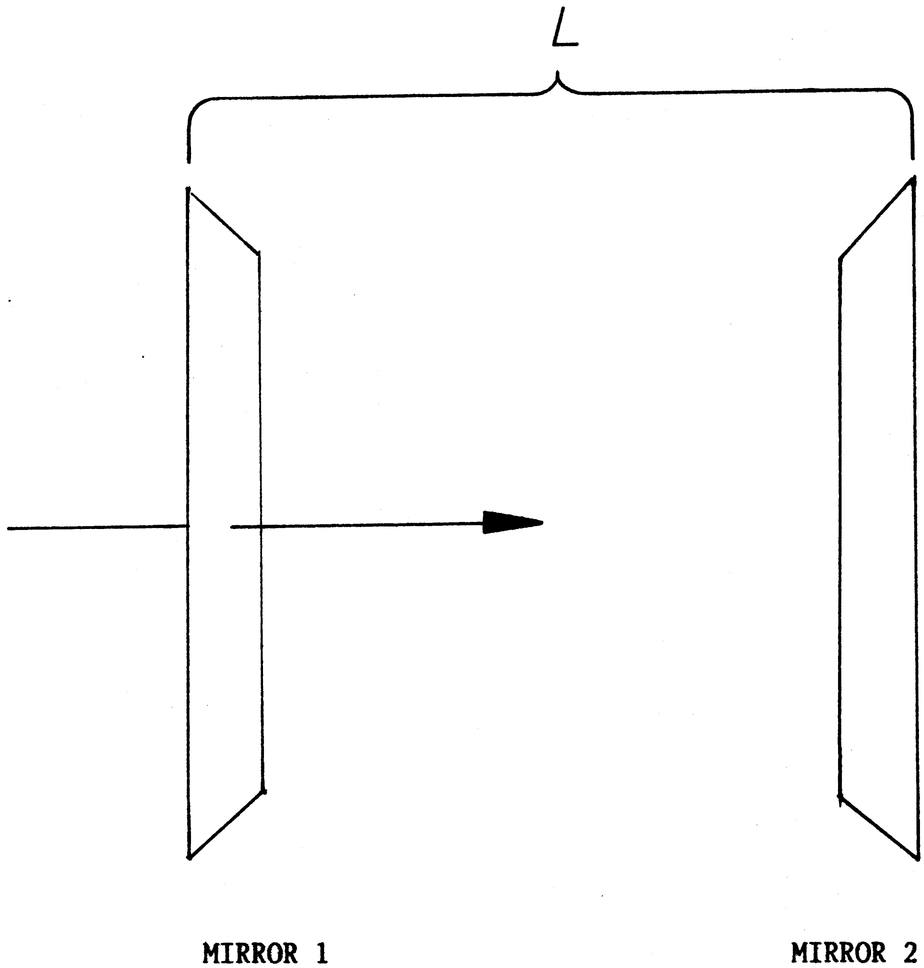

Figure 1.1. Perspective View of the Fabry-Perot Interferometer

Like all realizable filters, the passband-stopband boundary is not abrupt, but forms a smooth transition. In addition, the light energy introduced into the interferometer is attenuated based upon its wavelength/frequency composition characteristics, as well as the degree of reflectivity of the plane mirrors employed.

Referring to Figure 1, a Fabry-Perot interferometer consists of a mirror 1 and a mirror 2 oriented in a mutually parallel fashion, separated by a distance L. Light falling on mirror 1, propagating in the direction of mirror 2, will be partially reflected back to the left and partially transmitted into the L width cavity between mirrors 1 and 2. The light within the cavity propagating to the right then falls upon mirror 2, which is partially reflected back toward mirror 1, and partially transmitted through and to the right of mirror 2. The light propagating back toward mirror 1 is again partially reflected back into the interstitial cavity to the right and partially transmitted back through and away from mirror 1 to the left. The light in the interstitial space between mirror 1 and mirror 2 continues to partially reflect within and partially transmit through the mirrors to the outside.

The "mirrors" of Figure 1 are idealized as very thin mirrors. This is especially true for the treatment herein. By the use of the term "mirror," a partially reflecting and partially transmitting surface is indicated. Such a mirror may have a reflectivity, the percentage amplitude of light reflected to total light impinging a surface, of from 0 to 1, but preferably somewhere in between.

The mirrors commonly encountered have a mirrored surface applied to a thick dielectric. The thick dielectric, usually glass, is for the purpose of support. It will be clear from the following discussion that a "thick" dielectric surface mirror would form its own smaller Fabry-Perot interferometer, thus complicating the problem, and its ease of understanding.

However, a thin surface is achievable. In some cases a polished mirrored surface can be applied to the end of a fiber optic cable. The material applied to the dielectric surface can be by vapor deposition, and would be only a few molecular layers thick. Such a deposition technique allows quantitatively tighter control over the reflectivity.

In the case of a reflective material applied to a clear dielectric, the change in propagation medium from a clear dielectric to an air cavity would cause a very small error. Propagation from a clear dielectric to a dielectric cavity, of course, would cause, the least error. Utilization of a dielectric or other solid material in the interstitial cavity would probably severely limit design of the cavity housing.

But it is the geometry of and materials of construction of the housing around the cavity which is of great interest in utilizing the Fabry-Perot interferometer as a sensor. The parameter sensed is usually the dimension L, the separation between the mirrors. Depending upon the construction, Fabry-Perot interferometers can be made to sense temperature and pressure. The Fabry-Perot is most fully modeled as a two port, flow through filter where a signal is introduced at the first port and detected as it comes out of the second port. Given the real limitations of sensor installation, it would be optimum to have a sensor at the end of a single fiber rather than having a sensor located at the center a loop formed by the transmitter fiber, the sensor, and the receiver fiber. In such a single fiber system, mirror 2 would be totally reflecting with respect to the interstitial cavity, no light would enter the cavity from the outside through mirror 2, and the single fiber would form the conduit for light energy both to and from the Fabry-Perot sensor.

The Fresnel Formula

The Fresnel formulae define the basic interaction between light and a relatively transparent dielectric boundary. When an electromagnetic wave impinges a dielectric boundary, part of wave continues through the material, and part of the wave is reflected. Due to the continuity of the electric, or E field at the boundary, the reflected wave must have its magnetic, or H field reversed. This is because Maxwell's equations dictate that for an electric field pointing in one direction while propagating in second direction fixes the direction of the magnetic field exactly.

Figure 2 illustrates the relative orthagonal relationship between the electric field, E, and the magnetic field H, as it approaches a dielectric surface. The electric, or E field is shown to be continuous at a boundary. For a wave emanates from the boundary in the opposite direction, given that the E field is continuous at the boundary, the H field must be considered to change its orientation to maintain consistency with the coordinate system.

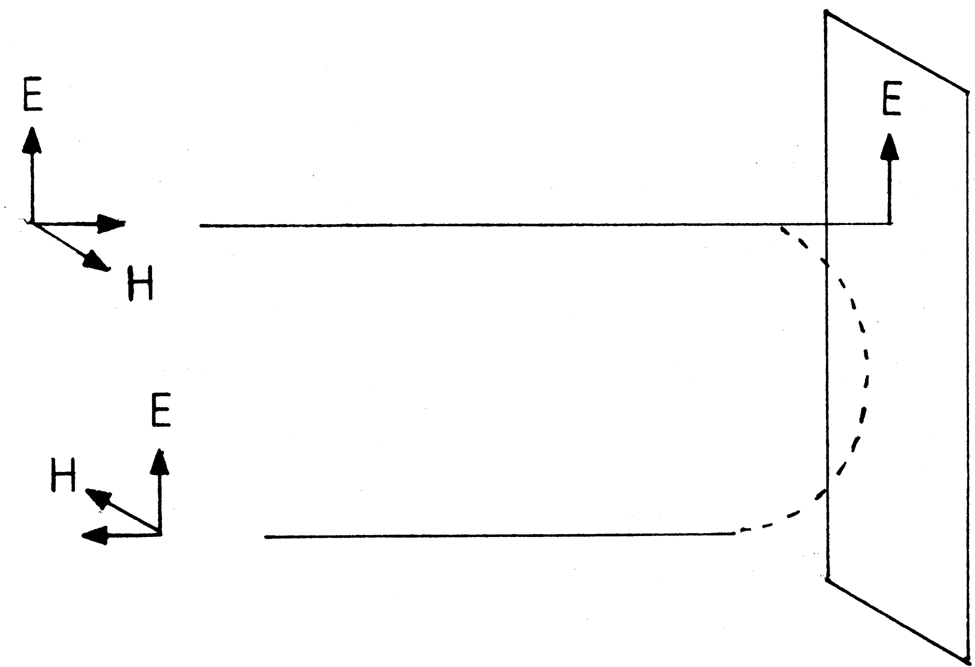

Figure 1.2. Electromagnetic Fields at a Dielectric Boundary

As a result of the above relationships, two important equations result:

$$E_x = E^+e^{-az}e^{-jBz} + E^-e^{+az}e^{+jBz} \qquad (1.1)$$

$$H_y = H^+e^{-az}e^{-jBz} - H^-e^{+az}e^{+jBz} \qquad (1.2)$$

In equations (1.1) and (1.2), E_x is the x component of the E field, Hy is the y component of the H field associated with E_x, and where the exponents relate to propagation in the positive z direction for a negative exponent and propagation in the negative z direction for a positive exponent.

These equations lead to a visually more identifiable expression:

$$E_i + E_r = E_t \qquad (1.3)$$

$$H_i - H_r = H_t \qquad (1.4)$$

Here the subscripts stand for the incident, reflected and transmitted fields. Given that the intrinsic impedance N for any region is the ratio of the E_x field divided by the H_y field, and using the second of the above equations, with the incident wave in zone 1 and with the transmitted wave in zone 2, yields the following:

$$N_1 = \frac{E_{xi}}{H_{yi}} \qquad (1.5) \qquad\qquad N_2 = \frac{E_{xt}}{H_{yt}} \qquad (1.6)$$

$$H_{yi} = \frac{E_{xi}}{N_1} \qquad (1.7) \qquad\qquad H_{yt} = \frac{E_{xt}}{N_2} \qquad (1.8)$$

Utilizing equations (1.7) and (1.8) into (1.4) above yields

$$\frac{E_{xi}}{N_1} - \frac{E_{xr}}{N_1} = \frac{E_{xt}}{N_2} \qquad (1.9)$$

Solving for the reflection coefficient yields:

$$r = \frac{E_r}{E_i} = \frac{N_2 - N_1}{N_1 + N_2} \qquad (1.10)$$

Similarly, solving for the transmission coefficient, t, yields:

$$t = \frac{E_t}{E_i} = \frac{2N_2}{N_1 + N_2} \qquad (1.11)$$

These are the most recognizable forms of this relationship. From equation 1.10, it is evident that the sign of the reflection coefficient is either positive or negative depending upon whether the electromagnetic wave propagates from a less dense (higher N) to a more dense (lower N) region or from a more dense to a less dense region.

Looking at it another way, for an incident wave having a phase of +1, the reflected wave could have a phase of +1 or -1, depending upon whether or not the propagation was from a region of given density into a relatively more dense or a relatively less dense region.

Remember that "t" and "r" are not the energies of the light transmitted and reflected, but represent the amplitudes of the light. For example, a light wave having an amplitude of 1 travelling in glass would have an amplitude of greater than 1 if it traversed the glass-air boundary and continued its propagation into the air region. This is evident from equation (1.11) when N_2 has a value in air of about 377, and when N_1 has a value in a particular glass of about 251.

However, "r" and "t" are related to energy. The square of "r" yields the percent energy reflected. Unity, or 1, minus the value of "t" squared yields the percent energy transmitted. Therefore, a matrix allows handling of amplitudes, phases, and, with the above relationships, energy.

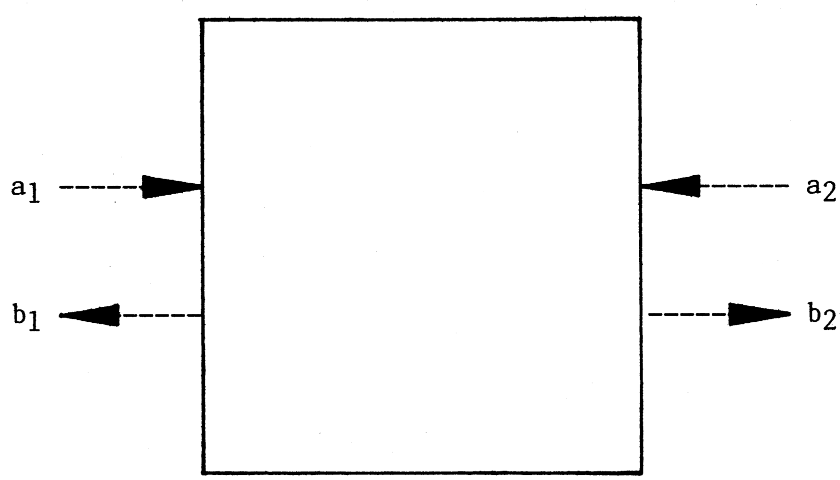

Figure 2.1. Generalized Scattering Matrix Representation

CHAPTER 2

SCATTERING PARAMETERS GENERALLY

The relationship between a plane mirror and optical energy lends itself to analysis based upon the use of scattering parameters. As is shown in Figure 2.1, a box defines a network, or some spatial entity. Energy can flow into the box or exit the box from the left hand side, port 1, and energy can flow into the box or exit the box from the right hand side, port 2. A mixed designation is utilized wherein the a_1 represents energy into port 1, b_1 represents energy out of port 1. Similarly, a_2 represents energy into port 2, and b_2 represents energy out of port 2.

The matrix is usually written in the following form:

$$\begin{matrix} b_1 \\ b_2 \end{matrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{matrix} a_1 \\ a_2 \end{matrix} \tag{2.1}$$

Where:

$$\underset{(a_2=0)}{S_{11}} = \frac{b_1}{a_1} \qquad \underset{(a_1=0)}{S_{22}} = \frac{b_2}{a_2}$$

$$\underset{(a_1=0)}{S_{12}} = \frac{b_1}{a_2} \qquad \underset{(a_2=0)}{S_{21}} = \frac{b_2}{a_1}$$

Therefore, each "cell" or position in the matrix governs a relationship between a different two of the entering and leaving amplitude relationships for the two port. The matrix representation lends better lends itself to a make sense description of what is going on in the two port system. This is especially true since the matrix operates upon inputs to produce outputs. It is easy to see how individual inputs could be masked or omitted, and the effect such omission has on the output quantities.

The letters a and b represent the amplitude of the electromagnetic wave entering and leaving the two port network, respectively. The "1" subscript indicates the left side boundary and the "2" subscript indicates the right side boundary. The energy within a wave is equal to the square of the amplitude. The square of any of the scattering matrix coefficients yields the corresponding energy in the wave. This relation parallels the scattering matrix elements relationship to energy. The square of a matrix element will yield the portion of energy reflected or the complement from unity of the portion of the energy transmitted.

Fabry-Perot Scattering Considerations

Figure 2.2 illustrates the Fabry-Perot system existing within the two mirrors which were previously shown in Figure 1. Figure 2.2, utilizing a zig-zag representation of the multiple light reflections, shows how each light beam has a portion of its energy reflected within and a portion transmitted outside of the mirror boundaries.

In a physical sense, as shown in Figure 2.2, a light wave entering from the left is partially reflected back to the source by

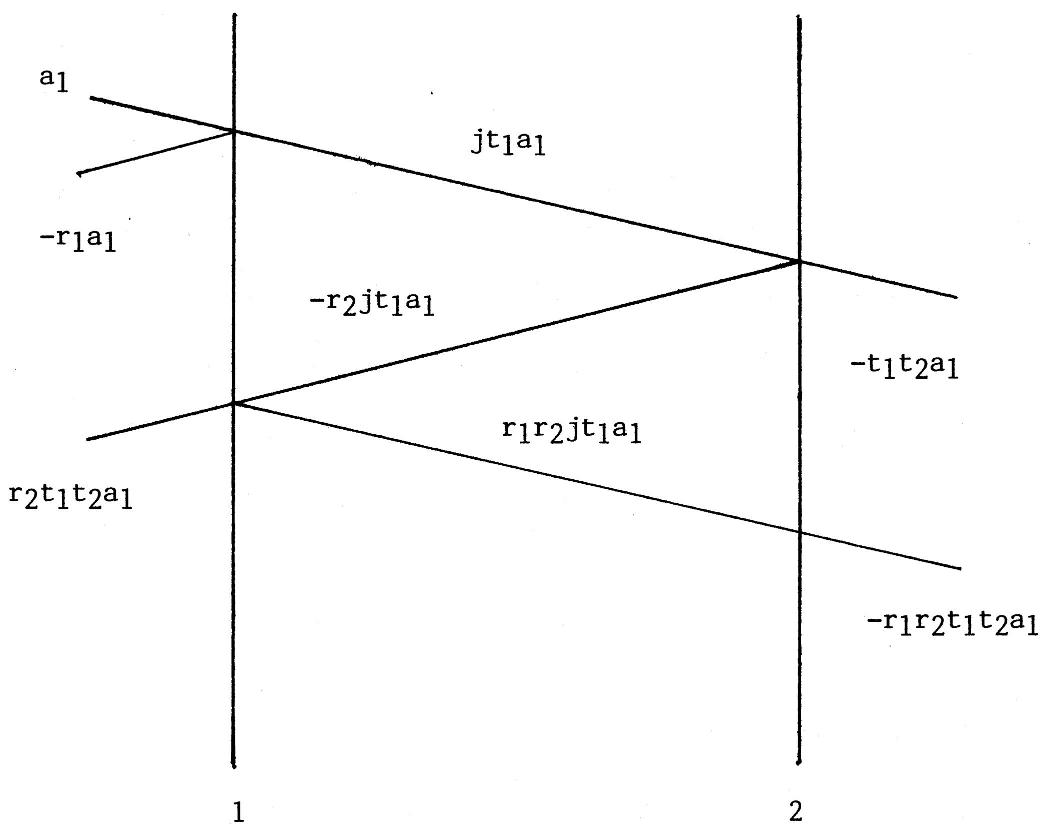

Figure 2.2. Diagram of Multiple Fabry-Perot Reflection

mirror 1. The part which is not reflected is transmitted into the middle or interstitial volume between mirrors 1 and 2. As the transmitted wave continues to the right, it arrives at mirror 2. Here, similar to the case of the approaching wave with respect to mirror 1, part of the light wave is reflected back in the direction of mirror 1 while part is transmitted through and to the right of mirror 2.

The light wave reflected back in the direction of mirror 1 is again partially reflected back in the direction of mirror 2, and partially transmitted back through mirror 1. This process, involving each residual reflected light wave in the interstitial area, continues in an infinite manner. Intuitively, the amount of light reflectively propagating between mirrors 1 and 2 will depend upon the reflectivity of the two boundaries.

Considering a beam of light entering the arrangement of Figure 3 from the left, the amount of total light propagating between the mirrors can be calculated from the infinite series resulting from the infinite numbers of reflections resulting between the mirrors. This quantity is key to further analysis of the two mirror system. For illustration only, Figure 2.2 shows a zig-zag line to enable view of subsequent internal reflections. Although such a system could be utilized and computed using the law of cosines to extract the horizontal portion of the wave, further analysis will be based upon an in line, here shown as horizontal, system to obviate the need for angular considerations. Next, rules for treating the reflective and transmissive propagation of an electromagnetic wave are examined.

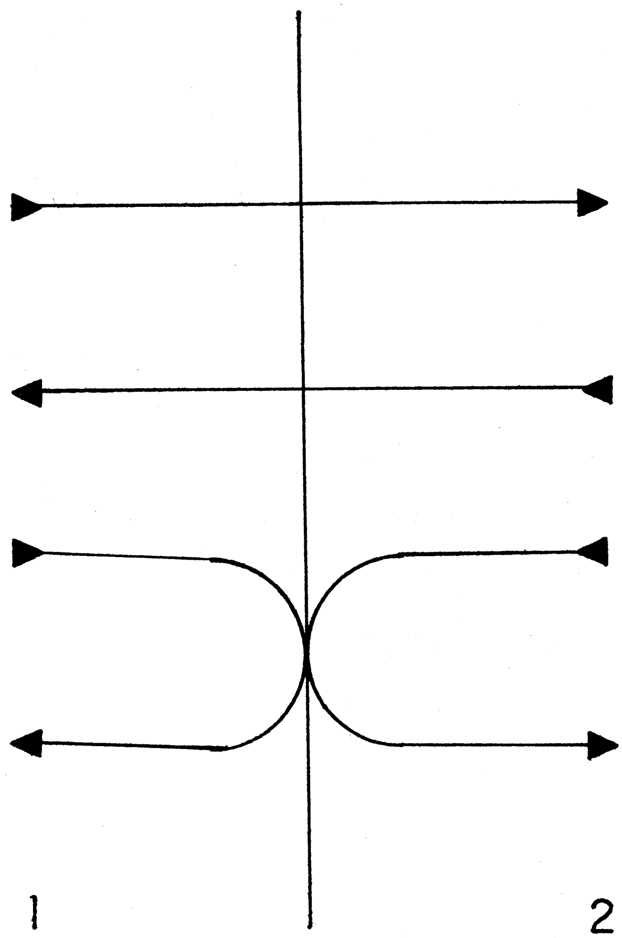

Figure 2.3. Scattering in a Thin Dielectric

The Scattering Matrix for a Thin Dielectric

Before analysis of the Fabry-Perot system, the characterization of what occurs regarding a light wave on reflection from and transmission through a mirror, and propagation through space, will need some explanation. Several properties may be assumed which will govern these characteristics, including reciprocity and conservation of energy. If mirrors 1 and 2 are both identically flat on both sides, each mirror will react to approaching light energy from one direction exactly just as it would to another direction.

In the specific case of light propagating in air toward a dielectric surface, the reflection coefficient gamma will be negative. The amount of mirroring on the surface of the dielectric will control the magnitude of the reflection coefficient, regardless of the side from which the surface of the dielectric is approached.

For a single mirror, a two port analysis would have S_{12} represent the energy in region 1 due to energy inputs from region 2, while S_{21} represents the energy in region 2 due to input from region 1. Based upon experience and arbitrary reversibility of any given mirror, it is intuitive that $S_{12} = S_{21}$. Similarly S_{11} represents the energy in region 1 due to energy inputs from region 1, while S_{22} represents the energy in region 2 due to input from region 2. Again reversibility of any given mirror leads to the observation that $S_{11} = S_{22}$.

As was previously mentioned, the "amplitude" of the light wave is related to the power in that the square of the absolute value of the amplitude represents the power. Of the power incident from the left, the transmitted and reflected power must add to 100% of that power. Similarly, of the power incident from the right, the transmitted and reflected power must add to 100% of that power. This is also true for any two port system which does not store energy. Stated in terms of the scattering parameters, the following equations result:

$$S_{11} \quad + \quad S_{21} \quad = \quad 1 \tag{2.2}$$

$$S_{22} \quad + \quad S_{12} \quad = \quad 1 \tag{2.3}$$

In addition, for a dissipationless network, the absolute value of the product of the members of each column must equal zero. Stated another way,

$$S_{11} \; S_{12} \; + \; S_{21} \; S_{22} \; = \; 0. \tag{2.4}$$

Since the absolute value of one number times another number can be can be defined as one of the numbers times the complex conjugate of the other number, the above equation (15) can be written:

$$S_{11}^{*} \; S_{12} \; + \; S_{21}^{*} \; S_{22} \; = \; 0. \tag{2.5}$$

This leads to the observation that

$$S_{12} = -S_{21}^{*} \frac{S_{22}}{S_{11}^{*}} \tag{2.6}$$

Since $S_{11} = S_{22}$, then $S_{12} = -S_{21}^{*}$. The only way for this relationship to be true is for S_{12} and S_{21} to be purely complex quantities, that is to be preceded by the complex operator j. S_{12} is the light wave in region 1 from and due to region 2 and S_{21} is the light wave in region 2 from and due to
region 1. Thus S_{12} and S_{21} deal with the transmission of light across the mirror boundary and will have magnitudes equal to the transmission coefficient t, or in the case as here, where t is complex, will have magnitudes equal to jt.

For reflectivity, since the tangential electric fields must match at the interface, the phase shift is 180 degrees, causing a sign change upon reflection. S_{11} and S_{22}, the reflection parameters of the scattering matrix are each equal to $-r_1$ and $-r_2$, respectively.

This results in a generalized scattering matrix for any single, very thin mirror, as follows:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \left[\begin{array}{c|c} -r & jt \\ \hline jt & -r \end{array}\right] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{2.7}$$

This results in a generalized scattering matrix for any single, very thin mirror, and can be assumed to be valid for any two port.

The Scattering Matrix for a Volume of Space

Insofar as propagation is concerned, a light wave changes phase as it propagates, as does any other travelling wave. For propagation normal to a frame of reference, the phase shift continues according to the relationship:

$$d = \frac{2\pi f n l}{c} = \frac{2\pi n l}{\lambda} \qquad (2.8)$$

where f is the frequency, c is the speed of light in a vacuum, n is an integer, l is the length traveled, and d is the shorthand designation for facilitating the representation of the exponential quantity. If l is the spacing between the mirrors then 2d represents the total round trip distance. The repeatability of this function is emphasized by the integer n. The phase of a light wave propagating within the system is more fully represented by the quantity

$$e^{-j2\pi f n l/c} = e^{-jd} \qquad (2.9)$$

It is increasingly clear that there is an optimum mirror spacing, or path length for a given wavelength of light. Conversely, broadband light will have certain of its frequency components selectively transmitted or attenuated through the mirrors of the Fabry-Perot interferometer based on the relationship of the frequency to mirror separation. Figure 2.4 illustrates this spatial relationship.

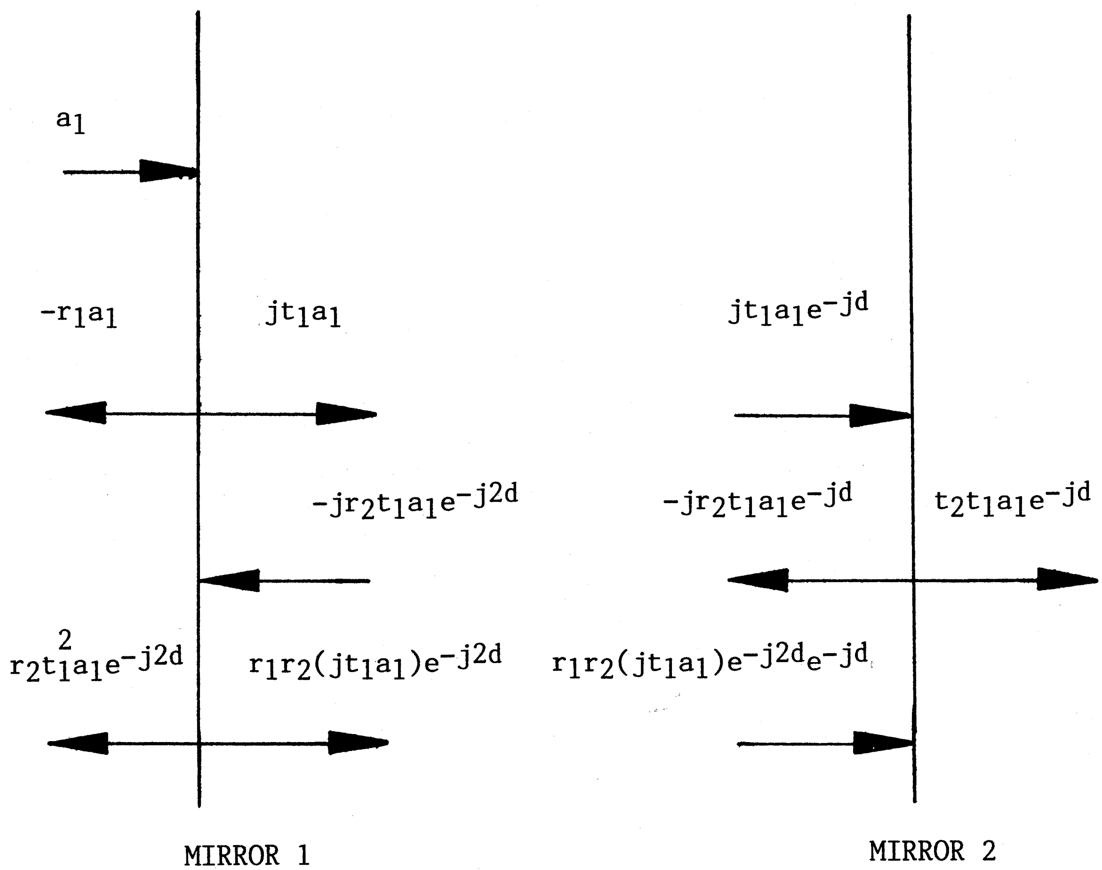

Figure 2.4. Stepwise Representation of Subsequent Reflections

The Fabry-Perot Scattering Matrix

Referring to Figure 2.4, the two mirror system of Figure 2.2 is illustrated without the potentially confusing angles. Each arrow level represents a single directional event for a light beam. By illustrating it this way, Figure 2.4 can be taken step at a time illustrating the multiple reflections of a single light beam without the confusing angular zig-zags used earlier to show multiple reflections. In this way both the reduction in amplitude due to multiple reflections, as well as the change in phase due to spatial propagation may be illustrated. Although the mechanism of multiple reflection and transmission would occur with a light beam of a non-normal angle between two dielectrics, a Fabry-Perot will typically utilize light at normal incidence.

Each subsequent level represents a different point in time of the life of the beam due to reflectivity from a surface due to the action of the beam above it. As is shown in Figure 2.4, light entering from the left is designated as a_1, this designation indicative of the "amplitude" of the light wave. Light leaving the system and propagating to the left is designated b_1. Light leaving the system at the right hand boundary is designated b_2, while light entering the system from the right is designated a_2. This symbols used are in general agreement with standard scattering parameter notation.

As previously discussed, light reflecting from a surface has its amplitude and phase multiplied by -r, the reflectivity coefficient for that surface representing a diminution in amplitude and a 180 degree phase change represented by the negative sign. Light transmitted through a surface has its amplitude and phase multiplied by jt, the transmission coefficient for that surface representing either an increase or a diminution in amplitude and the j indicative of a 90 degree phase change.

In Figure 2.4, all arrows pointing to the left of and away from mirror 1 contribute to b_1 while all arrows pointing to the right of and away from mirror 2 contribute to b_2. At the upper left hand corner, a beam of light represented by a_1 enters from the left and impinges upon mirror 1. This beam of light is partially reflected away from and to the left of mirror 1 and multiplied by a factor $-r_1$. The resulting leftward propagating light beam has a total amplitude equal to $-r_1 a_1$.

The other portion of this light beam a_1 reaching the interstitial space between the mirrors, and at a point **just** to the right of mirror 1, is multiplied by jt_1 and has a total amplitude represented by the product $jt_1 a_1$. As the light beam travels from a point just to the right of mirror 1 to a point just to the left of mirror 2, a phase change occurs. This phase change is dependent upon the length of separation between mirrors 1 and 2. This phase change is represented by a multiplicative factor of e^{-jd}. The results in an expression for the light wave at this point euqal to the multiplicative product of $jt_1 a_1 e^{-jd}$.

Once the light wave impinges upon mirror 2, part of the light wave is transmitted through mirror 2, leaving to the right of Figure 5, and takes on the additional factor jt_2. The resulting expression for this part of b_2 is $jt_2jt_1a_1e^{-jd}$, which, utilizing the relationship $j^2 = -1$, reduces to $-t_2t_1a_1e^{-jd}$. The portion of the light wave reflected from mirror 2 beginning its propagation back to mirror 1 takes on the additional factor of $-r_2$. The expression for this portion of the internal wave just to the left of mirror 2 is $-jr_2(t_1)a_1e^{-jd}$. As the $-jr_2t_1a_1e^{-jd}$ light wave propagates to the left, toward mirror 1, a phase change occurs by the factor e^{-jd}. The multiplicative addition of this factor yields an expression for the light wave at a point just to the right of mirror 1 equal to $-jr_2t_1a_1e^{-2jd}$.

Repeating the process once mirror 1 is reached, once the light wave impinges upon mirror 1, part of the light wave is transmitted through mirror 1, leaving to the left of Figure 2.4, and takes on the additional factor jt_1. The resulting expression for this part of b_1 is $-j^2r_2(t_1)^2a_1e^{-2jd}$, which reduces to $r_2(t_1)^2a_1e^{-2jd}$. The portion of the light wave reflected from mirror 1 beginning its propagation back to mirror 2 takes on the additional factor of $-r_1$. The expression for this portion of the internal wave just to the left of mirror 1 is $jr_1r_2t_1a_1e^{-2jd}$. For clarity this may be written as $r_1r_2(jt_1a_1)e^{-2jd}$ to illustrate that but for the left hand r_1r_2 factor and the right hand e^{-2jd} factor the kernel jt_1a_1 is the same as the previous starting point just to the right of mirror 1. An additional multiplicative factor of $r_1r_2e^{-2jd}$ will repeat for each round trip, reducing the magnitude of each successive term. Since light approaching mirror 2 acts essentially the same, the same result would apply for a light wave a_2 entering from the left.

Even though each subsequent level represents a different point in time of the life of the beam due to reflectivity from a surface due to the action of the beam above it, it would be helpful to treat the Fabry-Perot interferometer in steady state, taking to account all subsequent reflections. Since multiples of this factor will be multiplicatively factored into the expression for each round trip, and it continues to do so on a steady state basis, a steady-state summation of the factors can be formulated.

The same set of reflectivities and distance phase shifts are multiplicatively included as an addition to the total number of multiplicative products in the expression. By grouping each recurring set separately, a summation, describing each light beam in figure 2.4 can be formulated.

However, to insure that a complete round trip for each term is precisely included, the reference plane which was utilized above, and which was infinitesimally to the right of mirror 1, is selected. The total internal light beam, which is the sum of all internal light beams propagating **to the right** is:

$$a = \sum_{m=0}^{\infty} (r_1 \, r_2 \, e^{-j2d})^m \, jt_1 \, a_1 \qquad (2.10)$$

where a_1 is the amplitude of the light beam incident upon the system from the left. This expression is **exactly** equal to the following:

$$a = \frac{j\, t_1\, a_1}{1 - r_1 r_2 e^{-j2d}} \qquad (2.11)$$

The proof for this exact equivalency can be verified by computer or from any number of math handbooks. With the total interstitial "a" field determined, all other fields are determinable with reference thereto. The total internal "b" or leftward propagating field will consist of the total internal "a" field times the reflection coefficient at mirror 2, namely r_2, along with the appropriate phase factor for the reference position. The interstitial "b" field, again at a reference point just to the right of mirror 1, is as follows:

$$b = \frac{j t_1 r_2 e^{-j2d} a_1}{1 - r_1 r_2 e^{-j2d}} \qquad (2.12)$$

All of the other quantities can now be easily computed with respect to equations (2.11) and (2.12). Note for emphasis that equations (2.11) and (2.12) are the "a" and "b" fields **internal** with respect to the Fabry-Perot interferometer, and they are taken not everywhere, but at a point just to the right of mirror 1 as previously specified. The **external** fields are dependent upon the internal fields **as well as reflection** from the external surfaces.

Next, to construct the scattering matrix for the Fabry-Perot system, we will analyze a light beam entering at the left, and its effect on b_1 and b_2. The first contribution to b_1 comes from the reflectivity of the outside of mirror 1, namely $-r_1a_1$. Light then entering the interstitial space between mirrors 1 and 2 is jt_1a_1, and it proceeds to form the multiple reflections referred to above.

For simplicity it is advantageous to refer to the total interstitial "b" field due to multiple reflections at the same reference point, namely just infinitesimally to the right of mirror 1. The portion of the "b" field which will propagate back through mirror 1 will be multiplied by the "through glass" scattering parameter referred to above as jt_1.

So, multiplicatively combining the interstitial b field with jt_1, and additively combining this quantity with the wave reflected from the surface of mirror 1 will yield:

$$b_1 = -r_1 a_1 + \frac{t_1^2 r_2 e^{-j2d}}{1 - r_1 r_2 e^{-j2d}} a_1 \qquad (2.13)$$

Recall that all reflections from mirror 2 in the direction of region 1 were taken to account in the computation of the total interstitial fields. Continuing, to further simplify this result, the first term on the right hand side of the equation is multiplied in the numerator and denominator by the denominator of the interstitial "b" field to yield:

$$b_1 = \frac{-r_1 a_1 + r_1^2 r_2 a_1 e^{-j2d} + t_1^2 r_2 a_1 e^{-j2d}}{1 - r_1 r_2 e^{-j2d}} \qquad (2.14)$$

Recognizing the common term for the two rightmost quantities in the numerator yields:

$$b_1 = \frac{-r_1 a_1 + r_2 a_1 e^{-j2d} (r_1^2 + t_1^2)}{1 - r_1 r_2 e^{-j2d}} \qquad (2.15)$$

Since the square of the reflectivity indicates reflected energy, and the square of the transmissivity indicates transmitted energy, and since the reflected and transmitted energy must sum to unity for the same reference plane, i.e. $r^2 + t^2 = 1$, we then have:

$$b_1 = \frac{-r_1 a_1 + r_2 a_1 e^{-j2d}}{1 - r_1 r_2 e^{-j2d}} \qquad (2.16)$$

Further, extracting the a_1 term will better illustrate the matrix term:

$$b_1 = \frac{-r_1 + r_2 e^{-j2d}}{1 - r_1 r_2 e^{-j2d}} \, a_1 \qquad (2.17)$$

The above term to the left of the a_1 term occupies the S_{11} position in the scattering matrix since it describes the contribution to b_1 from the a_1 input.

Similarly, the b_2 term as a function of a_1 can be computed from the internal "a" field of equation (2.11). As discussed above, and beginning just to the right of mirror 1, a factor of e^{-jd} will be multiplied to reach mirror 2, and a factor of jt_2 for transmission through mirror 2, will be applied to the total internal "a" field of equation (2.11). This yields:

$$b_2 = \frac{j\, t_1\, a_1\, jt_2\, e^{-jd}}{1 - r_1 r_2 e^{-j2d}} \tag{2.18}$$

The j x j factor forms a -1, and, extracting the a_1 term as was done above, reduces equation (2.18) to:

$$b_2 = - \frac{t_1\, t_2\, e^{-jd}}{1 - r_1 r_2 e^{-j2d}} a_1 \tag{2.19}$$

The portion of the equation to the left of a_1 forms element S_{21} of the scattering matrix, the portion of b_2 dependent upon a_1.

The remaining portions of the scattering matrix, including S_{12} and S_{22}, involve those portions of b_1 and b_2 attributable to inputs at a_2. Similar to the equations beginning with equation (2.11) and (2.12), and especially since the Fabry-Perot works the same, no matter from which side approached, two new equations can be written. These two new equations, to show the similarity between the computations above for S_{11} and S_{21} and those now made for S_{12} and S_{22}, will chose a reference point just to the left side of mirror 2. The quantities referenced to that point will carry a "'" primed designation. The "a'" field is still propagating to the right and the "b'" field is still propagating to the left.

$$b' = \frac{j\, t_2\, a_2}{1 - r_1 r_2 e^{-j2d}} \tag{2.20}$$

With the total interstitial "b'" field determined, all other fields are determinable with reference thereto. The total internal "a'" or rightward propagating field will consist of the total internal "b'" field times the reflection coefficient at mirror 1, namely r_1, along with the appropriate phase factor for the reference position. The interstitial "a'" field, again at a reference point just to the left of mirror 2, is as follows:

$$a' = \frac{j t_2\, r_1\, e^{-j2d}\, a_2}{1 - r_1 r_2 e^{-j2d}} \tag{2.21}$$

The portion of b_2 attributable to a_2, using (31) on the right side is:

$$b_2 = -r_2 a_2 + \frac{t_2^2 r_1 e^{-j2d}}{1 - r_1 r_2 e^{-j2d}} a_2 \tag{2.22}$$

Again continuing, to further simplify this result, the first term on the right hand side of the equation is multiplied in the numerator and denominator by the denominator of the interstitial "a'" field to yield:

$$b_2 = \frac{-r_2 a_2 + r_2^2 r_1 a_2 e^{-j2d} + t_2^2 r_1 a_2 e^{-j2d}}{1 - r_1 r_2 e^{-j2d}} \tag{2.23}$$

Again recognizing the common term for the two rightmost quantities in the numerator yields:

$$b_2 = \frac{-r_2 a_2 + r_1 a_2 e^{-j2d} (r_2^2 + t_2^2)}{1 - r_1 r_2 e^{-j2d}} \tag{2.24}$$

Again, since the square of the reflectivity indicates reflected energy, and the square of the transmissivity indicates transmitted energy, and since the reflected and transmitted energy must sum to unity for the same reference plane, i.e. $r^2 + t^2 = 1$, we then have:

$$b_2 = \frac{-r_2 a_2 + r_1 a_2 e^{-j2d}}{1 - r_1 r_2 e^{-j2d}} \tag{2.25}$$

Again, extracting the a_2 term will better illustrate the matrix term:

$$b_2 = \frac{-r_2 + r_1 e^{-j2d}}{1 - r_1 r_2 e^{-j2d}} \quad a_2 \tag{2.26}$$

The above term to the left of the a_1 term occupies the S_{22} position in the scattering matrix since it describes the contribution to b_2 from the a_2 input.

Similarly, the b_1 term of a function of a_2 can be computed from the internal "b'" field of equation (2.20). As discussed above, since we, in the latter case begin just to the left of mirror 2, the factor of e^{-jd} will be multiplied to the quantity in equation (2.20). Only an additional factor of jt_1 for transmission through mirror 1, will need be applied to the total internal "b'" field of equation (2.20). This yields:

$$b_1 = \frac{j\, t_1\, a_2\, jt_2\, e^{-jd}}{1 - r_1 r_2 e^{-j2d}} \tag{2.27}$$

The j x j factor forms a -1, and, extracting the a_2 term as was done above, reduces equation (37) to:

$$b_1 = - \frac{t_1\, t_2\, e^{-jd}}{1 - r_1 r_2 e^{-j2d}} \quad a_2 \tag{2.28}$$

This equation forms element S_{12} of the scattering matrix, the portion of b_1 dependent upon a_2. **The full scattering matrix for the Fabry-Perot**, collecting equations (2.17), (2.19), (2.26), and (2.28), **is:**

$$\begin{bmatrix} b_1 \\ \\ b_2 \end{bmatrix} = \begin{bmatrix} \dfrac{-r_1 + r_2 e^{-j2d}}{1 - r_1 r_2 e^{-j2d}} & -\dfrac{t_1 t_2 e^{-jd}}{1 - r_1 r_2 e^{-j2d}} \\ \\ -\dfrac{t_1 t_2 e^{-jd}}{1 - r_1 r_2 e^{-j2d}} & \dfrac{-r_2 + r_1 e^{-j2d}}{1 - r_1 r_2 e^{-j2d}} \end{bmatrix} \begin{bmatrix} a_1 \\ \\ a_2 \end{bmatrix} \qquad (2.29)$$

At first blush, a sign difference between S_{11} and S_{22} indicated, but the r_1 and r_2 positions are physically simply reversed. Since light approaching mirror 2 is reflected and contributes $-r_2a_2$ to the b_2 quantity, the equivalency of the above equation is seen.

Although each mirror is considered to be infinitely thin, and that in a physical realization of a Fabry-Perot system with thicker mirrors, the mirrors themselves form resonant cavities. However, the resonant system will be dominated by the interplay between the actual mirror surfaces. Since the mirrors are considered to be infinitely thin, the values of t, the transmission coefficient will always be less than unity since the light wave is never considered to be propagating through a volume of lesser impedance/greater optical density such as glass, into a volume of greater impedance/lesser optical density such as air. Therefore, each additional multiplicative term diminishes the result.

CHAPTER 3

THE TRANSMISSION MATRIX GENERALLY

The order of the elements within a transmission matrix differs from that of a scattering matrix. This is to enable the transmission matrices to be multiplicatively joined to describe a system.

$$\begin{bmatrix} b_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} b_2 \\ a_2 \end{bmatrix} \tag{3.1}$$

Compare this matrix system with the matrix of equation 2.1:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{2.1}$$

In equation (3.1), the input/output interface, say interface 2, of a two port system in terms of its input a_2 and its output b_2 can be operated upon by the transmission matrix of equation (3.1) to yield a "new" or updated input/output interface 1 with a next adjacent input/output pair, namely b_2 and a_2. Many transmission matrices may be sequentially multiplied against an initial output/input pair b_2/a_2 to yield a sequential series of intermediate output pairs. In this way the system states are known at each point.

Alternatively, the matrices may be multiplied together to form a large matrix which may then be multiplied against an initial output/input pair b_2/a_2 to yield the final output b_1/a_1 pair. This is especially useful where the intermediate states are not required to be known. A larger matrix utilized in a single matrix multiplication can be performed more rapidly, conserving computer time.

The order of multiplication of the matrix is important. Reversing the order of the matrices will cause a non-sense multiplication of the elements. Even though several matrices may be multiplied to make a single larger matrix, each multiplication step still "processes" the input to the system to several intermediate stages, the larger matrix utilizable to perform the whole "process" at one time.

Although the transmission matrix also has a "sense" orientation in that each component represents a contribution from identifiable sources, and as mentioned above, the transmission matrix does not lend itself readily to an experimental or logical manipulation in isolating the sources of the outputs. T_{11}, for example represents the portion of energy in b_1 due to a_2. T_{12}, similarly represents the portion of energy b_1 due to b_2, a relationship which is not directly logically apparent. **However**, note that the simple identity relationships cannot be in a real sense stated for each transmission coefficient as was the case for the scattering matrix. This is because the dependencies are neither logical, nor physically isolatable.

In the scattering matrix equations, the **input** to a two port system could be blocked off to allow the testing of one of the outputs in terms of one of the inputs. The blocking off of an input is a physically realizable phenomenon. For the transmission matrix, the definition of T_{11} would involve the "shutting off" of a two port output.

The shutting off of an output in a physically realizable sense cannot be accomplished easily, or without interfering with the system in some other way, or without introducing a "new" system portion to complicate the process. The same holds true for the other transmission matrix members. This is why the transformation from scattering matrices, with their "make sense" and physically realizable elements, to transmission matrices is so important and valuable.

Most obviously, note that scattering matrices give the outputs in terms of the inputs. The transmission matrices give the input/output pair for a single port by operating on the input/output pair of the next adjacent port. The ability to compute a transmission matrix from a scattering matrix, especially for matrices as large as equation (39) and larger can facilitate the analysis of a system, particularly the Fabry-Perot.

To derive the transmission matrix from the scattering matrix, the equation formed from row one of scattering matrix 12 is first utilized.

$$b_1 = S_{11} a_1 + S_{12} a_2 \qquad (3.2)$$

Solving for a_2 yields

$$a_2 = \frac{b_1}{S_{12}} - \frac{S_{11}\, a_1}{S_{12}} \qquad (3.3)$$

Utilizing the equation formed from row two of scattering matrix equation 2.1, namely

$$b_2 = S_{21}\, a_1 + S_{22}\, a_2 \qquad (3.4)$$

yields the following expression:

$$b_2 = S_{21}\, a_1 + \frac{S_{22}\, b_1}{S_{12}} - \frac{S_{22}\, S_{11}\, a_1}{S_{12}} \qquad (3.5)$$

Next, multiply the first term of equation 3.5 by S_{12}/S_{12}:

$$b_2 = \frac{S_{12}\, S_{21}\, a_1}{S_{12}} + \frac{S_{22}\, b_1}{S_{12}} - \frac{S_{22}\, S_{11}\, a_1}{S_{12}} \qquad (3.6)$$

Next, collect the terms having a common a_1 factor.

$$b_2 = \frac{(S_{12}\, S_{21} - S_{22}\, S_{11})\, a_1}{S_{12}} + \frac{S_{22}\, b_1}{S_{12}} \qquad (3.7)$$

The two right hand terms of equation (3.7) form the T_{11} and T_{12} elements of the transmission matrix. To find the T_{21} and T_{22} elements of the transmission matrix, next a_2 is computed in terms of a_1 and b_1 utilizing the relationship of equation 3.2.

$$b_1 = S_{11} a_1 + S_{12} a_2 \tag{3.2}$$

Equation 31 can be directly manipulated to yield

$$a_2 = \frac{-S_{11} a_1}{S_{12}} + \frac{b_1}{S_{12}} \tag{3.8}$$

By collecting the solutions for b_2 and a_2 in equations (3.7) and (3.8), the solutions as ordered above exactly fit the transmission matrix format in terms of both the matrix operated upon and in terms of the generated matrix.

$$\begin{bmatrix} b_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{(S_{12} S_{21} - S_{22} S_{11})}{S_{12}} & \frac{S_{22}}{S_{12}} \\ \frac{-S_{11}}{S_{12}} & \frac{1}{S_{12}} \end{bmatrix} \begin{bmatrix} b_1 \\ a_1 \end{bmatrix} \tag{3.9}$$

The Serial Joining of Transmission Matrices

As previously stated, transmission matrices for a system may be serially joined to yield the total system output. The large composite matrix gives the output for a given input, but a serial treatment of the smallest matrix unit on a given output sequentially yields the intermediate states of the energy in process. The next equation illustrates a "chain" of smaller transmission matrices.

$$\begin{bmatrix} b_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}_z \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}_y \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}_x \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}_w \begin{bmatrix} b_2 \\ a_2 \end{bmatrix} \quad (3.10)$$

Each matrix w, x, y, and z, in sequence, treats an initial input from port 1. Note that the flow of the signal or energy will be from port 2, through device w, x, y, and finally z. It may appear that the matrix of equation (3.10) was written backwards, but it is written in the order of multiplication. Matrix w is multiplied against the b_2 a_2 output/input matrix, then matrix x is multiplied against the result, then matrix y, and finally matrix z is multiplied against that result to form the final result. As a check, smaller transmission matrices formed by manipulation of the small scattering matrices should form the same larger transmission matrix formed by manipulation of the large scattering matrix.

Small Element Matrix Transformation

In equation (2.7) the scattering matrix for a thin dielectric was set forth. Equation (3.9) puts forward the transformation equation for transforming a scattering matrix to a transmission matrix.

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \left[\begin{array}{cc} -r & jt \\ \hline jt & -r \end{array}\right] \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad (2.7)$$

$$\begin{bmatrix} b_2 \\ \\ a_2 \end{bmatrix} = \begin{bmatrix} \dfrac{(S_{12}\,S_{21} - S_{22}\,S_{11})}{S_{12}} & \dfrac{S_{22}}{S_{12}} \\ \\ \dfrac{-S_{11}}{S_{12}} & \dfrac{1}{S_{12}} \end{bmatrix} \begin{bmatrix} b_1 \\ \\ a_1 \end{bmatrix} \tag{3.9}$$

Inserting the elements of (2.7) into (3.9), for a single mirror, say mirror 1, yields the transmission matrix for a thin dielectric:

$$\begin{bmatrix} b_2 \\ \\ a_2 \end{bmatrix} = \begin{bmatrix} \dfrac{(jt_1 jt_1 - ((-r_1)(-r_1)))}{jt_1} & \dfrac{-r_1}{jt_1} \\ \\ \dfrac{r_1}{jt_1} & \dfrac{1}{jt_1} \end{bmatrix} \begin{bmatrix} b_1 \\ \\ a_1 \end{bmatrix} \tag{3.11}$$

Reducing further yields:

$$\begin{bmatrix} b_2 \\ \\ a_2 \end{bmatrix} = \begin{bmatrix} \dfrac{(-t_1^2 - r_1^2)}{jt_1} & \dfrac{-r_1}{jt_1} \\ \\ \dfrac{r_1}{jt_1} & \dfrac{1}{jt_1} \end{bmatrix} \begin{bmatrix} b_1 \\ \\ a_1 \end{bmatrix} \qquad (3.12)$$

Since $t^2 = 1 - r^2$, T_{11}, the upper left term becomes unity, which then yields:

$$\begin{bmatrix} b_2 \\ \\ a_2 \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{jt_1} & \dfrac{-r_1}{jt_1} \\ \\ \dfrac{r_1}{jt_1} & \dfrac{1}{jt_1} \end{bmatrix} \begin{bmatrix} b_1 \\ \\ a_1 \end{bmatrix} \qquad (3.13)$$

Similarly for mirror 2, equation (3.13) becomes:

$$\begin{bmatrix} b_2 \\ \\ a_2 \end{bmatrix} = \begin{bmatrix} \dfrac{-1}{jt_2} & \dfrac{-r_2}{jt_2} \\ \\ \dfrac{r_2}{jt_2} & \dfrac{1}{jt_2} \end{bmatrix} \begin{bmatrix} b_1 \\ \\ a_1 \end{bmatrix} \tag{3.14}$$

The relationship between phase and space described in equation (2.9) can be stated in terms of a scattering matrix should be transformable to a scattering matrix by the application of simple logic. As recited above in equation (2.1), the matrix is usually written in the following form:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{2.1}$$

Consider the generalized scattering matrix at Figure 3 to be a section of free space having a left boundary at port 1 and a right boundary at port 2. For this section of free space, **there is no reflection at the space boundary.** Light entering from the left, a_1, continues through the volume within the boundaries of the free space section boundary, emerging as b_2. Likewise, light entering from the right, a_2, continues through the volume of space within the boundaries of the free space section boundary and emerges as b_1.

Since 100% of the amplitude is transmitted straight through, this sets the relationship between a_1 and b_2, and between a_2 and b_1 as unity. This corresponds to the S_{21} and the S_{12} entries, respectively, of the scattering matrix shown in equation (2.1). However, during the propagation through the space between the two ports, a phase change occurs. As stated in equation (2.9), this phase change is equivalent to e^{-jd}, which has an absolute value of unity.

Similarly, there is no reflection at the boundaries meaning that no beam a_1 ever becomes or contributes to b_1. No beam a_2 ever becomes or contributes to b_2. This means that the absolute value of the relationships between a_1 and b_1, and between a_2 and b_2 is **zero.** This corresponds to the S_{11} and the S_{22} entries, respectively, of the scattering matrix shown in equation (2.1).

The following matrix results:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 & e^{-jd} \\ e^{-jd} & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{3.15}$$

Next, utilizing equation (3.9) for transformation from a scattering to transmission matrix, the following matrix results:

$$\begin{bmatrix} b_2 \\ a_2 \end{bmatrix} = \begin{bmatrix} \dfrac{(e^{-jd}\, e^{-jd} - (0)\,(0))}{e^{-jd}} & \dfrac{0}{e^{-jd}} \\ \dfrac{0}{e^{-jd}} & \dfrac{1}{e^{-jd}} \end{bmatrix} \begin{bmatrix} b_1 \\ a_1 \end{bmatrix} \tag{3.16}$$

Reducing further yields:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} e^{-jd} & 0 \\ 0 & e^{+jd} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{3.17}$$

which is the expression for a transmission matrix for a section of free space.

The Fabry-Perot Transmission Matrix

Again taking equation (3.9), equation (2.29) is inserted therein. First, for ease of explanation, the factor $1/(1-r_1r_2e^{-j2d})$ is factored out of equation (2.29).

$$\frac{1}{1 - r_1 r_2 e^{-j2d}} \begin{bmatrix} -r_1 + r_2 e^{-j2d} & -t_1 t_2 e^{-jd} \\ -t_1 t_2 e^{-jd} & -r_2 + r_1 e^{-j2d} \end{bmatrix} \tag{3.18}$$

Plugging the matrix portion of equation (3.18) into equation (3.9) will require that the left side factor of equation (3.18) be included. To facilitate this, equation (3.9) is again illustrated, and the computations will be performed one element at a time.

$$\begin{bmatrix} b_2 \\ \\ a_2 \end{bmatrix} = \begin{bmatrix} \dfrac{(S_{12}\, S_{21} - S_{22}\, S_{11})}{S_{12}} & \dfrac{S_{22}}{S_{12}} \\ \\ \dfrac{-S_{11}}{S_{12}} & \dfrac{1}{S_{12}} \end{bmatrix} \begin{bmatrix} b_1 \\ \\ a_1 \end{bmatrix} \tag{3.9}$$

The whole of equation (3.18) into equation (3.9) is the following matrix:

$$\begin{bmatrix} \dfrac{\dfrac{(t_2)^2(t_1)^2e^{-j2d}}{(1 - r_1r_2\, e^{-j2d})^2} - \dfrac{((r_1r_2)-((r_1)^2e^{-j2d})-((r_2)^2e^{-j2d})+(r_1r_2e^{-j4d})}{(1 - r_1r_2\, e^{-j2d})^2}}{\dfrac{-\,t_1\, t_2\, e^{-jd}}{(1 - r_1r_2\, e^{-j2d})}} & \dfrac{\dfrac{r_2-r_1e^{-j2d}}{(1-r_1r_2e^{-j2d})}}{\dfrac{t_1\, t_2\, e^{-jd}}{(1-r_1r_2\, e^{-j2d})}} \\ \\ \dfrac{\dfrac{-r1 + r2\, e^{-j2d}}{(1 - r_1r_2\, e^{-j2d})}}{\dfrac{t_1\, t_2\, e^{-jd}}{(1 - r_1r_2\, e^{-j2d})}} & \dfrac{-\,1}{\dfrac{t_1\, t_2\, e^{-jd}}{(1-r_1r_2\, e^{-j2d})}} \end{bmatrix}$$

(3.18A)

The upper left, or T_{11} element of the transformation gives:

$$\frac{\dfrac{(t_2)^2(t_1)^2e^{-j2d}}{(1 - r_1r_2\ e^{-j2d})^2} - \dfrac{((r_1r_2)-((r_1)^2e^{-j2d})-((r_2)^2e^{-j2d})+(r_1r_{2e}{}^{-j4d})}{(1 - r_1r_2\ e^{-j2d})^2}}{\dfrac{-\ t_1\ t_2\ e^{-jd}}{(1 - r_1r_2\ e^{-j2d})}} \tag{3.19}$$

Dividing the large common factors yields:

$$\frac{(t_2)^2(t_1)^2e^{-j2d}}{(1-r_1r_2e^{-j2d})(-t_1t_2\ e^{-jd})} - \frac{((r_1r_2)-((r_1)^2e^{-j2d})-((r_2)^2e^{-j2d})+(r_1r_{2e}{}^{-j4d})}{(1 - r_1r_2\ e^{-j2d})(-t_1t_{2e}{}^{-jd})} \tag{3.20}$$

Both terms have the same common denominator. In addition, to enable cancellation, the t terms in the numerator will be substituted for by using the relationship $t^2 = 1 - r^2$. This will put the final result in terms of the reflection coefficients.

$$\frac{(1-r_2^2)(1-r_1^2)e^{-j2d} - ((r_1r_2)-((r_1)^2e^{-j2d})-((r_2)^2e^{-j2d})+(r_1r_{2e}{}^{-j4d})}{(1 - r_1r_2\ e^{-j2d})(-t_1t_{2e}{}^{-jd})} \tag{3.21}$$

Next, the term in the left half of the numerator is expanded, and the parenthetical expressions in the upper right half of the numerator are eliminated with the appropriate sign assignments.

$$\frac{(1 - r_2^2 - r_1^2 + r_2^2 r_1^2)e^{-j2d} - r_1 r_2 + r_1^2 e^{-j2d} + r_2^2 e^{-j2d} - r_1 r_2 e^{-j4d}}{(1 - r_1 r_2\, e^{-j2d})(-t_1 t_2 e^{-jd})} \tag{3.22}$$

Noting the two sets of like terms, and performing the cancellation gives:

$$\frac{e^{-j2d} + r_2 r_1\, e^{-j2d} - r_1 r_2 - r_1 r_2 e^{-j4d}}{(1 - r_1 r_2\, e^{-j2d})(-t_1 t_2 e^{-jd})} \tag{3.23}$$

Noting that the numerator can be factored into two multiplicative factors, one of which matches the denominator yields:

$$\frac{(e^{-j2d} - r_1 r_2)\ (1 - r_1 r_2\, e^{-j2d})}{(1 - r_1 r_2\, e^{-j2d})(-t_1 t_2 e^{-jd})} \tag{3.24}$$

Cancelling:

$$\frac{(e^{-j2d} - r_1 r_2)}{(-t_1 t_2 e^{-jd})} \tag{3.25}$$

Dividing by e^{-jd} yields and moving the bottom negative sign:

$$\frac{(-e^{-jd} + r_1 r_2 e^{+jd})}{(t_1 t_2)} \tag{3.26}$$

Next the upper right, or T_{12} element is computed

$$\frac{\dfrac{r_2 - r_1 e^{-j2d}}{(1 - r_1 r_2 e^{-j2d})}}{\dfrac{t_1 \; t_2 \; e^{-jd}}{(1 - r_1 r_2 \; e^{-j2d})}} \tag{3.27}$$

Cancelling like denominators, and dividing by e^{-jd} gives:

$$\frac{r_2 e^{+jd} - r_1 e^{-jd}}{t_1 \; t_2} \tag{3.28}$$

Next, the lower left, or T_{21} element of the transmission matrix is treated:

$$\frac{\dfrac{-r1 + r2 \quad e^{-j2d}}{(1 - r_1 r_2 \; e^{-j2d})}}{\dfrac{t_1 \; t_2 \; e^{-jd}}{(1 - r_1 r_2 \; e^{-j2d})}} \tag{3.29}$$

Cancelling like denominators, and dividing by e^{-jd} gives:

$$\frac{-r1\ e^{+jd} + r2\ e^{-jd}}{t_1\ t_2} \tag{3.30}$$

Finally, the lower right transmission element, T_{22} is:

$$\frac{-\ 1}{\dfrac{t_1\ t_2\ e^{-jd}}{(1 - r_1 r_2\ e^{-j2d})}} \tag{3.31}$$

Inverting and negativing the denominator into the numerator gives:

$$\frac{(r_1 r_2\ e^{-j2d} - 1)}{t_1\ t_2\ e^{-jd}} \tag{3.32}$$

Dividing by e^{-jd} gives:

$$\frac{(r_1 r_2\ e^{-jd} - e^{+jd})}{t_1\ t_2} \tag{3.33}$$

Combining all of the elements of equations (3.26), (3.28), (3.30), and (3.33) of the transmission matrix for the Fabry-Perot yields:

$$\begin{bmatrix} \dfrac{(-e^{-jd} + r_1 r_2 e^{+jd})}{t_1\ t_2} & \dfrac{(r_2 e^{+jd} - r_1 e^{-jd})}{t_1\ t_2} \\ \dfrac{-r1\ e^{+jd} + r2\ e^{-jd}}{t_1\ t_2} & \dfrac{(r_1 r_2\ e^{-jd} - e^{+jd})}{t_1\ t_2} \end{bmatrix} \tag{3.34}$$

In order to visualize how the reflectivity, which of course determines the transmissivity, affects the transmission matrix, the final set of t's may be stated in terms of their respective r's.

$$\begin{bmatrix} \dfrac{(-e^{-jd} + r_1 r_2 e^{+jd})}{((1-r_1^2)(1-r_2^2))^{.5}} & \dfrac{(r_2 e^{+jd} - r_1 e^{-jd})}{((1-r_1^2)(1-r_2^2))^{.5}} \\ \dfrac{-r1\ e^{+jd} + r2\ e^{-jd}}{((1-r_1^2)(1-r_2^2))^{.5}} & \dfrac{(r_1 r_2\ e^{-jd} - e^{+jd})}{((1-r_1^2)(1-r_2^2))^{.5}} \end{bmatrix} \tag{3.35}$$

As a check, the individual transmission matrices (3.13), (3.14), and (3.17) can be multiplied together to yield equation (3.35). Although the Fabry-Perot can accept light into either port, the matrices will be arranged to treat port 1 as an input since it will be less confusing when the single fiber Fabry-Perot is considered.

$$\begin{bmatrix} b_2 \\ \\ a_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{-1}{jt_2} & \dfrac{-r_2}{jt_2} \\ \dfrac{r_2}{jt_2} & \dfrac{1}{jt_2} \end{bmatrix}}_{(3.13)} \underbrace{\begin{bmatrix} e^{-jd} & 0 \\ 0 & e^{+jd} \end{bmatrix}}_{(3.14)} \underbrace{\begin{bmatrix} \dfrac{-1}{jt_1} & \dfrac{-r_1}{jt_1} \\ \dfrac{r_1}{jt_1} & \dfrac{1}{jt_1} \end{bmatrix}}_{(3.17)} \begin{bmatrix} b_1 \\ \\ a_1 \end{bmatrix} \quad (3.36)$$

$$\begin{bmatrix} b_2 \\ \\ a_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{-1}{jt_2} & \dfrac{-r_2}{jt_2} \\ \dfrac{r_1}{jt_2} & \dfrac{1}{jt_2} \end{bmatrix}}_{(3.13)} \underbrace{\begin{bmatrix} \dfrac{-e^{-jd}}{jt_1} & \dfrac{-r_1e^{-jd}}{jt_1} \\ \dfrac{r_1e^{+jd}}{jt_1} & \dfrac{e^{+jd}}{jt_1} \end{bmatrix}}_{\text{(intermediate)}} \begin{bmatrix} b_1 \\ \\ a_1 \end{bmatrix} \quad (3.37)$$

$$\begin{bmatrix} b_2 \\ \\ a_2 \end{bmatrix} = \begin{bmatrix} \dfrac{e^{-jd} - r_1r_2e^{+jd}}{-\ t_1\ t_2} & \dfrac{+r_1e^{-jd} - r_2e^{+jd}}{-\ t_1\ t_2} \\ \dfrac{-r_2\ e^{-jd} + r_1e^{+jd}}{-\ t_1\ t_2} & \dfrac{-r_1r_2e^{-jd} + e^{+jd}}{-\ t_1\ t_2} \end{bmatrix} \begin{bmatrix} b_1 \\ \\ a_1 \end{bmatrix} \quad (3.37)$$

The negative signs are due to the j^2 factor. Consolidating signs:

$$\begin{bmatrix} b_2 \\ \\ a_2 \end{bmatrix} = \begin{bmatrix} \dfrac{-e^{-jd} + r_1r_2e^{+jd}}{t_1\ t_2} & \dfrac{-r_1e^{-jd} + r_2e^{+jd}}{t_1\ t_2} \\ \dfrac{r_2\ e^{-jd} - r_1e^{+jd}}{t_1\ t_2} & \dfrac{r_1r_2e^{-jd} - e^{+jd}}{t_1\ t_2} \end{bmatrix} \begin{bmatrix} b_1 \\ \\ a_1 \end{bmatrix} \quad (3.38)$$

Note that equation (76) agrees with equation (71), and that the scattering matrix for the Fabry-Perot is derivable directly, or from the individual transmission matrices.

CHAPTER 4

SINGLE OPTICAL FIBER OPERATION

After considering the parameters outlined above, and considering a two mirror interferometer with a single fiber approach, a condition is necessary in order for the system to work.

To illustrate how a single fiber interferometer might be set up, first, consider a system in which the first mirror, mirror 1, has a reflection coefficient greater than zero and less than unity, and where mirror 2 has a reflection coefficient of unity. In terms of the total energy, it is clear that such a sensor returns 100% of the energy all the time. This is, of course, excluding losses. If 100% of the energy returns, the interferometer cannot operate in the standard Fabry-Perot mode. There would be no discernment between differing spacing of the mirrors.

Referring to equation (2.29), and considering a_2 to be zero since mirror r_2 is totally reflecting and no light energy may enter, and ignoring b_2, only the upper left term of the matrix is relevant:

$$b_1 = \frac{-r_1 + r_2 \, e^{-j2d}}{1 - r_1 r_2 \, e^{-j2d}} \, a_1 \tag{4.1}$$

The r_2 term is unity. Dividing by a_1 to get a ratio yields:

$$\frac{b_1}{a_1} = \frac{-r_1 + e^{-j2d}}{1 - r_1 e^{-j2d}} \tag{4.2}$$

A cursory analysis of equation (4.2) indicates that where 2d is equal to 2 , the ratio is one exactly. For half that value, the ratio is equal to negative one. For /4, the ratio varies from 1 $\underline{/0}$ degrees for r_1 equal to zero to 1 $\underline{/180}$ degrees for r_1 equal to unity. Thus all of the energy is returned, and the only difference is in the phase of the light returning. Due to the coherence length of light used, the average "length" or time duration of a single photon would make measuring the phase difference virtually impossible. Each photon, or a composite of all photons would have to be phase measured upon transmission and return. Such a system is not conducive to simple, steady-state operation.

One way to preserve the Fabry-Perot mechanism is to allow mirror 2 to be partially reflective, to allow energy to escape the interferometer beyond mirror 2. Such an escape would need to be guarded, such that no random light energy entered mirror 2. This would entail covering the surface of mirror 2 with a light absorbing material, or surrounding mirror 2 with a non-reflective surface. The surface would additionally need to be in a position to dissipate heat. Instead of using the light transmitted through and beyond mirror 2 as a measure of mirror separation, the light reflected back to the source would be utilized.

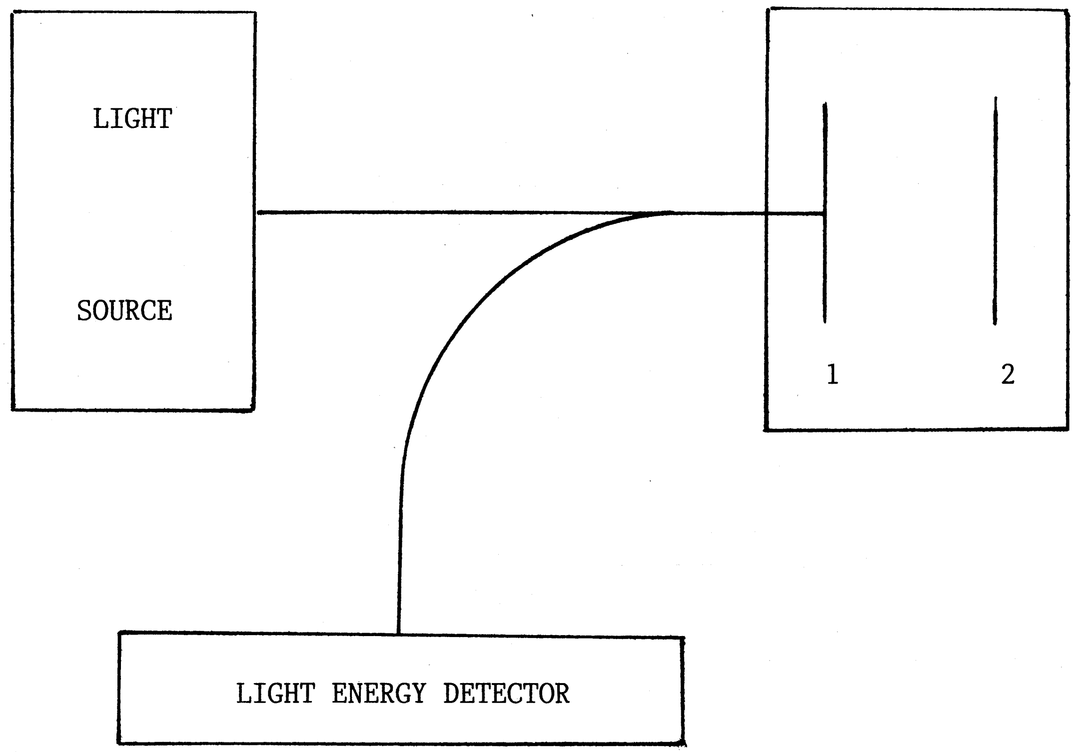

Figure 4.1. Single Fiber Interferometer Schematic

A directional coupler capable of coupling light propagating away from the interferometer could be used as an indicator of relative mirror position. Such a device is shown in schematic form in Figure 4.1 on the previous page. An alternate embodiment could include a colored mirror 2 along with a mirror 1 colored on the inside, both also partially reflective. Mirror 1 would be a color neutral mirror. The internal "a" field due to multiple reflections against mirrors 1 and 2 would then be of one color.

Instead of a simple fiber optic directional coupler, a wavelength division multiplexer could be used. For the standard Fabry-Perot, the energy return consists of both the reflection from the outside mirror 1, which is a constant, and would be color neutral and the reflections due to the internal field. The constant energy reflected due to mirror 1 retains a constant minimum energy difference regardless of the mirror spacing. This will increase the required sensitivity for the electronic detector utilized to detect changes in mirror spacing. The increased sensitivity will be necessary to detect the difference between the energy peaks and valleys. Light of a given color reflected by mirror 2 would build into an internal "b" field of a constant color. This constant color light, even though propagating back to the detector along with light of the original transmitted color, will be selectively split off by the wavelength division multiplexer, and thus be measured directly. In this manner, the color modified light can be measured in the absence of light reflecting from mirror 1, and the resolution of a given detector can be similarly increased.

Practical Considerations

In the optical fiber, all of the light does not propagate along its axial center in a straight line. Neither does all of the light travel in a straight line as it propagates along the fiber. Light "bounces" or reflects from side to side as it propagates through the fiber. As a result, at the "gap" between mirror 1 formed by the polished end of the fiber and mirror 2 formed by a planar reflective surface displaced by the fiber, not all of the light will propagate normal to mirrors' planar surfaces.

The portions of light which manage to leave mirror 1, travel to mirror 2 at an angle, reflect and re-enter mirror 1 will have traveled a longer path than the portion of the light which propagated between the mirrors at a normal angle. A greater path length favors a slightly lower frequency component. This deviation will tend to blur or make less sharp the boundary between the wavelengths of light whose propagation through the Fabry-Perot are favored versus those wavelengths of light whose propagation is suppressed.

The degree to which a non-normal portion of light affects the total result is, however, inversely proportional to the anglular deviation from normal incidence of the light leaving mirror 1. This is because a higher angle will cause a lesser number of multiple reflections between the mirrors 1 and 2 before the light wave veers to a point where it cannot be recaptured by the mirrors. For example, a small angle deviation might support several hundred internal reflections before veering off as a loss. A larger angle deviation might only allow 2 or 3 internal reflections before veering off as a loss. Therefore, for larger angles, the path length difference with respect to normal incidence will be much greater, but the effect of smaller numbers of internal multiple reflections will mitigate the effect. For smaller angles, the path length difference with respect to normal incidence will be much less, but the effect of larger numbers of internal multiple reflections will enhance the effect.

Truncating these two offsetting effects is a limitation known as the cone of acceptance. In order for light to propagate along a fiber optic line, the angle of propagation with respect to the internal wall of the fiber must be sufficiently small for total internal reflection to occur. Larger angles with respect to a line parallel to the walls cause losses. Due to Snell's law, the angle of incidence on the end of a fiber optic line determines the propagation angle within the fiber. An angle greater than this minimum angle represents a loss, and therefore a cutoff.

An estimate of the maximum percent deviation due to this non-ideality may be made once the maximum angle of acceptance and Fabry-Perot separation is known. In Figure 4.2, a section of fiber is shown. The fiber is made of a material having index of refraction n_1 and surrounded by a material (or no material) having an index of refraction n_2. The end of the fiber forms mirror 1, while mirror 2 is shown to the left. Angle a_1 is made with a line axially parallel to and outside the fiber. Angle a_2 is made with a line axially parallel to and within the fiber. Angle a_2 differs from angle a_1 due to Snell's law and the difference in refractive index between the

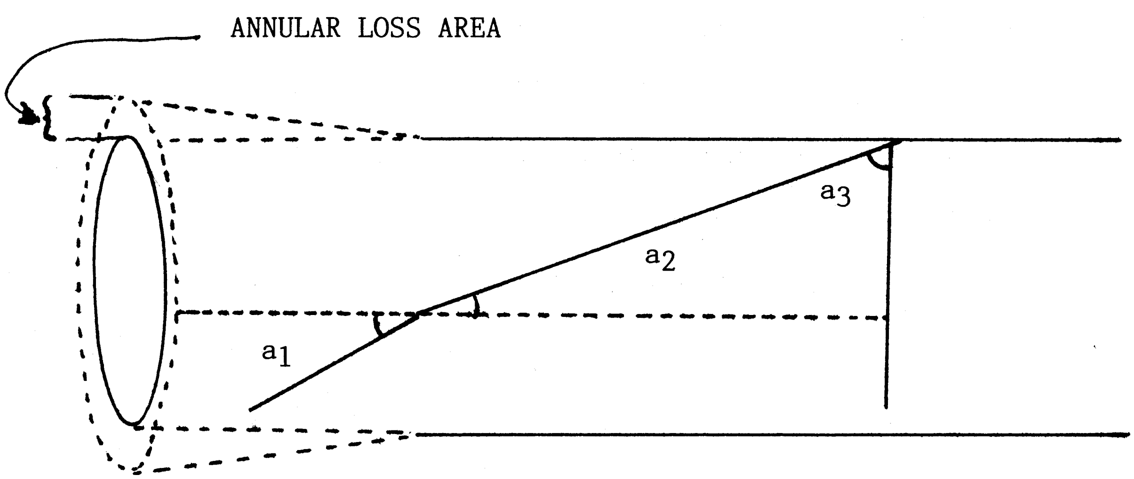

Figure 4.2. Cone of Acceptance Schematic

space between the mirrors and the inside of the fiber. An angle a_3 with respect to the inner wall of the fiber is 90 degrees different from the angle a_2.

The angle a_3 is the total internal reflectance angle, or the critical angle given by:

$$\text{Sin } a_3 = n_2/n_1 \tag{4.3}$$

The angle a_1 is related to angle a_2 by Snell's law, where the index of refraction for air is 1.0 and is:

$$1.0 \sin a_1 = n_1 \sin a_2 \tag{4.4}$$

As previously stated, the 90 degree difference between a_2 and a_3, in equation form is:

$$a_2 = 90 \text{ degrees} - a_3 \tag{4.5}$$

Using the relationship that $\cos^2 + \sin^2 = 1$, $\cos a_2 = \sin a_3 = n_2/n_1$, yields:

$$\sin a_1 = n_1{}^2 - n_2{}^2 \tag{4.6}$$

So, the index of refraction of the fiber and the material or lack thereof surrounding it determines the angle of acceptance a_1. As an example for illustrative purposes, consider a 100 micron diameter fiber has a polished end forming mirror 1 separated from a mirror 2 having an excessive diameter, by 500 nanometers. Consider the optical fiber to have a refractive index of $n_1 = 1.6$ and the outer cladding to have a refractive index $n_2 = 1.5$. According to the equation above, a_1 becomes 33.8 degrees. A 33.8 degree angle, over a base length of 500 nanometers forms a hypotenuse of 601 nanometers.

This means that light may enter the fiber at angles of deviation from normal of from 0 degrees experiencing a mirror spacing of 500 nanometers to 33.8 degrees experiencing a mirror spacing of 601 nanometers. This path difference is a 16.8% maximum relative pathlength difference. It is not plus or minus since the path length can increase, not decrease. Given that light should be uniform across an array of angles whose average is equal to 33.8 degrees, the average pathlength difference should see a 16.9 degree angle corresponding to a 522 nanometer hypotenuse for a right angle between the mirrors. This relative pathlength difference is 4.45%, about one fourth of the maximum calculated above.

The attenuation occurs both with respect to light leaving the fiber at an angle greater than 33.8 degrees and light attempting to enter which is greater than 33.8 degrees. The cone of acceptance would at first appear to dictate the size of mirror 2, but it must be noted that the worst case for reflection back into the fiber would be a photon leaving one edge of the fiber, or the polished edge of mirror 1, reflecting across the center. Obviously, any light from one edge propagating concentrically away from the center axis of the fiber will be lost in any event. However, light from one edge which propagates normally with respect to the mirrors should have the size of the mirror opposite of equal diameter to permit the return trip. Therefore, the size of mirror 1 and mirror 2 will be approximately equal with no significant increase in loss.

Given that the diameter of the fiber is relatively larger than the gap, an expansive area is available to accept light, and the angle of acceptance governs light entering at any point, not just near the center or edges. Moreover, remembering that light propagating down the fiber toward mirror 1 is subject to the same internal reflection angle limitation, it is clear that all of the light available for propagation between the mirrors should originate at an angle no greater than the 33.8 degree "cone of light". If this is the case, the loss can be computed by considering the 33.8 degree frusto-conical volume of light between the mirrors, and finding the ratio of mirror 1 to the area illuminated at the 500 nanometer separation, given the 33.8 degree "spread" which will be lost.

The sine of the 33.8 degree angle multiplied times the 601 nanometer hypotenuse of the right angle formed by the spread will give the additional radius of lost light via missed illumination of mirror 2. This represents an additional radius of 334.3 nanometers. For a fiber diameter of 100 microns, and assuming even illumination, the illuminated diameter at 500 nanometers becomes 100.668 microns. The percentage recaptured light is the square of the radius of the fiber divided by the square of the radius of the illuminated area. Here this quantity is $(50 \text{ microns})^2/(50.334 \text{ microns})^2$, or .9867. So, the efficiency is about 98.67%. Again, this assumes that the only light available for multiple reflection is within a 33.8 degree cone of light. Also, since the average angle should be less than 33.8 degrees, this efficiency is conservative.

The assumption that all light leaving the fiber is within this cone of light aids in introducing the last source of error, that of truncation of the infinite multiple reflections due to light waves angling off of the mirrors. For normal propagation, an infinite number of reflections is possible to a point where the quantum of energy necessary to support a photon's existence falls below its minimum threshold. Light, even if it enters from the edge of the fiber and propagates toward the center will re-reflect a finite, even if large, number of times.

In the above example, given an average angle of 16.9 degrees and a cross diameter travel of 44 nanometers per round trip reflection over the diameter of the 100 micron diameter fiber, the number of reflections would be the former length divided into the latter diameter. This equates to 2272 trips. Similarly the maximum angle of 33.8 degrees and a diameter travel of 668.6 nanometers per round trip equates to 149 round trips. An average 2272 reflections, and using a .5 reflectivity for the interferometer creates an error smaller than 10^{-99}. The 149 round trip case creates an error smaller than 10^{-89}. This error is negligible compared with illumination loss due to the cone of acceptance/ cone of illumination loss.

Conclusion

As the above computations indicate, the Fabry-Perot interferometer is readily characterizable in terms of its energy and phase. As is implied, the maximum use of this device may be had where devices and techniques capable of the resolution of small energy differences is available.

In addition, the above derivations, scattering matrices and transmission matrices are universal in that they do not relate to one particular material. Any material need only have its reflectivity and transmissivity characterized to be utilizable to form a Fabry-Perot interferometer. An understanding of such energy matrices aids in the physical understanding and predications of the performance of the device, inviting further experimentation and configuration building.